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A note on the classical BRST symmetry of the pure spinor string in a curved background

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Abstract: The classical pure spinor version of the heterotic superstring in a supergravity and super Yang-Mills background is considered. We obtain the BRST transformations of the world-sheet fields. They are consistent with the constraints obtained from the nilpotence of the BSRT charge and the holomorphicity of the BRST current.

Keywords: Conformal Field Models in String Theory, Superstrings and Heterotic Strings.

Contents

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1. Introduction

A covariant formalism of the superstring was formulated six years ago by Berkovits [1]. Since then, this formalism has passed many tests which include the calculation of tree-level [2] and higher loops [3] scattering amplitudes. It was also proven that the formalism describes correctly the superstring degrees of freedom, in fact the superstring spectrum was determined in the light-cone gauge [4] and in [5] it was constructed the first massive state in terms of a manifestly ten dimensional supercovariant language. Recently, Berkovits realized that his formalism admits a more geometrical origin by discovering a topological formulation [6].

The formalism can be adapted to describe strings in curved backgrounds including those with Ramond-Ramond fluxes like anti de Sitter spaces [7]. There, quantum conformal invariance [8] and quantum BRST invariance [9] have been verified.

Berkovits and Howe constructed the sigma model action suitable to describe ten dimensional supergravity backgrounds [10] (see also [11]). The sigma model action is the most general classically conformal invariant compatible with the isometries of the background. The classical BRST invariance of the model implies that the background fields are constrained to satisfy the ten dimensional supergravity equations of motion. In [12] it was shown that the conformal invariance is preserved in the quantum regime at the one-loop level if the background is constrained by classical BRST invariance. The next logical step is to preserve quantum BRST invariance to obtain α'-corrections in the supergravity equations of motion. In this calculation it will be useful to determine how the world-sheet fields transform under classical BRST transformations. The purpose of this paper is to determine such transformations.

In the next section we review the sigma model action for the heterotic string in the pure spinor formalism. In section 3 we derive the classical BRST transformations of the world-sheet fields.[1] In the final section we find consistency with the constraints of [10] derived from the nilpotence of the BRST charge and the holomorphicity of the BRST current.

2. The pure spinor approach to the heterotic superstring

Let us remind the sigma model action for the heterotic string in a background supporting gauge and gravitational fields, it is given by

$$S = \frac{1}{2\pi\alpha'}\int d^2z\Big[\frac{1}{2}\Pi^a\overline{\Pi}^b\eta_{ab} + \frac{1}{2}\Pi^A\overline{\Pi}^B B_{BA} + \Pi^A\overline{J}^I A_{IA} + d_\alpha(\overline{\Pi}^\alpha + \overline{J}^I W_I^\alpha) + \lambda^\alpha\omega_\beta(\overline{\Pi}^A\Omega_{A\alpha}{}^\beta + \overline{J}^I U_{I\alpha}{}^\beta)\Big] + S_{\overline{J}} + S_{\lambda,\omega} + S_{FT}, \qquad (2.1)$$

where $\Pi^A = \partial Z^M E_M{}^A, \overline{\Pi}^A = \overline{\partial} Z^M E_M{}^A$ with $E_M{}^A$ being the supervielbein, $Z^M = (x^m, \theta^\mu)$; $m = 0, \ldots, 9$, $\mu = 1, \ldots, 16$ the superspace coordinates, d_α the world-sheet generator of superspace translations. $S_{\overline{J}}$ is the action for the gauge group variables, $S_{\lambda,\omega}$ is the action for the pure spinor variables (λ, ω)[2] and S_{FT} is the Fradkin-Tseytlin which is proportional to $\int d^2zr\Phi$, where r is the world-sheet curvature and the θ-independent term of the superfield Φ is the dilaton. This term is not necessary when we study the classical dynamics of the system. However, it helps to restore the quantum conformal invariance as it was shown in [12]. The other background fields in (2.1) are the 2-form superfield B, the gauge field A_I, the superfields W_I^α (whose lowest component is the gaugino), $U_{I\alpha}{}^\beta$ (whose lowest component is related to the gauge field strength) and $\Omega_{A\alpha}{}^\beta$ is the Lorentz connection.

The action (2.1) has two local Lorentz transformations. One rotates the spinor indices as $\delta\lambda^\alpha = \lambda^\beta\Sigma_\beta{}^\alpha$ and the other rotates the vector indices as $\delta\Pi^a = \Pi^b\Lambda_b{}^a$. At the end of the day, both Lorentz transformations turn out to be identified, namely $tr(\gamma^{ab}\Sigma)$ is proportional to Λ^{ab}. The action is also invariant under gauge transformations of the gauge group SO(32) or $E_8 \times E_8$.

The quantization of the system given by (2.1) is performed by studying the cohomology of the BRST operator $Q_{\rm BRST} = \oint j_{\rm BRST} = \oint \lambda^\alpha d_\alpha$. As it was demonstrated in [10], the constraints on the backgrounds fields of the ten dimensional SUGRA/SYM system are implied by the nilpotence of the BRST charge and the holomorphicity of the BRST current. Let us reobtain this result in slightly different manner. We first determine how the world-sheet fields are transformed by the action of $Q_{\rm BRST}$. In order to do this, we define the canonical conjugate to Z^M as

$$P_M = (2\pi\alpha')\frac{\delta S}{\delta(\partial_0 Z^M)} \qquad (2.2)$$
$$= -E_M{}^\alpha d_\alpha + \frac{1}{2}(\Pi_a + \overline{\Pi}_a)E_M{}^a - \frac{1}{2}(\Pi^A - \overline{\Pi}^A)B_{AM} + \overline{J}^I A_{IM} + \lambda^\alpha\omega_\beta\Omega_{M\alpha}{}^\beta,$$

[1] These transformations are also obtained in [13].

[2] Since the pure spinor λ is constrained to satisfy $\lambda\gamma^a\lambda = 0$, its canonical conjugate ω is defined up to $\delta\omega_\alpha = (\gamma^a\lambda)_\alpha\Lambda_a$ for a parameter Λ. Then, $\lambda^\alpha\omega_\beta$ can only be expressed in terms of the ghost number current $J = \lambda^\alpha\omega_\alpha$ and the generator for pure spinors Lorentz rotations $N^{ab} = \frac{1}{2}(\lambda\gamma^{ab}\omega)$.

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where ∂_0 is respect to the world-sheet time σ^0. In this way we can relate P_M to the world-sheet field d_α. We also use the canonical commutation relations

$$[P_M, Z^N] = -\delta_M^N, \qquad [\lambda^\alpha, \omega_\beta] = \delta_\beta^\alpha, \qquad [\overline{J}^I, \overline{J}^J] = f^{IJ}{}_K \overline{J}^K, \tag{2.3}$$

where f's are structure constants of the gauge group. Note that these commutations relations are done at equal world-sheet times and that there is a delta function $\delta(\sigma^1 - \sigma'^1)$ in the r.h.s. of each.

As it was shown in [10], the nilpotence of $Q_{\rm BRST}$ can be computed after writing d_α in terms of the canonical variables and using the canonical commutations relations of (2.3). The holomorphicity of $j_{\rm BRST}$ is determined from the equations of motion derived from the action (2.1). In these ways, the background fields satisfy the nilpotence constraints

$$\lambda^\alpha \lambda^\beta T_{\alpha\beta}{}^A = \lambda^\alpha \lambda^\beta H_{\alpha\beta A} = \lambda^\alpha \lambda^\beta F_{I\alpha\beta} = \lambda^\alpha \lambda^\beta \lambda^\gamma R_{\alpha\beta\gamma}{}^\delta = 0, \tag{2.4}$$

and the holomorphicity constraints

$$T_{\alpha(ab)} = H_{\alpha ab} = T_{a\alpha}{}^\beta = \lambda^\alpha \lambda^\beta R_{a\alpha\beta}{}^\gamma = T_{\alpha\beta a} - H_{\alpha\beta a} = F_{Ia\alpha} - W_I^\beta T_{\alpha\beta a} = 0, \tag{2.5}$$
$$F_{I\alpha\beta} - \frac{1}{2} H_{\alpha\beta\gamma} W_I^\gamma = \nabla_\alpha W_I^\beta + W_I^\gamma T_{\gamma\alpha}{}^\beta - U_{I\alpha}{}^\beta = \lambda^\alpha \lambda^\beta (\nabla_\alpha U_{I\beta}{}^\gamma + R_{\delta\alpha\beta}{}^\gamma W_I^\delta) = 0,$$

where T, R, H and F are the torsion, the Lorentz curvature, the gauge field strength and the three-from curvature of the two-form B. In [10] it was proved that these constraints put the background fields on-shell, that they satisfy the N=1 D=10 SUGRA/SYM equations of motion.

3. BRST transformations of the world-sheet fields

We define the BRST transformation of a field Ψ as $\delta_B \Psi = [\oint \epsilon \lambda^\alpha d_\alpha, \Psi]$, where ϵ is a constant Grassmann number and the Poisson bracket is calculated from the canonical commutation relations of (2.3). To do this, we need to express the world-sheet field d_α in terms of the P_M and the other world-sheet fields. From the definition (2.2) one obtains

$$d_\alpha = -E_\alpha{}^M P_M - \frac{1}{2}(\Pi^A - \overline{\Pi}^A) B_{A\alpha} + \overline{J}^I A_{I\alpha} + \lambda^\beta \omega_\gamma \Omega_{\alpha\beta}{}^\gamma. \tag{3.1}$$

Now it will be shown that $\delta_B = \delta_g + \tilde{\delta}$, where δ_g refers to the gauge transformation with parameter $-\epsilon \lambda^\alpha A_{I\alpha}$ and a Lorentz transformation with parameter $-\epsilon \lambda^\gamma \Omega_{\gamma\beta}{}^\alpha$.

Consider first the pure spinor λ^α. We obtain

$$\delta_B \lambda^\alpha = \oint d\sigma' \epsilon \lambda^\beta(\sigma')[d_\beta(\sigma'), \lambda^\alpha(\sigma)] = \oint d\sigma' \epsilon \lambda^\beta(\sigma') \lambda^\gamma(\sigma') \Omega_{\beta\gamma}{}^\delta(\sigma')[\omega_\delta(\sigma'), \lambda^\alpha(\sigma)]$$
$$= \lambda^\beta(-\epsilon \lambda^\gamma \Omega_{\gamma\beta}{}^\alpha),$$

which corresponds to a Lorentz rotation of the pure spinor λ^α with parameter $-\epsilon \lambda^\gamma \Omega_{\gamma\beta}{}^\alpha$.

Consider now the conjugate pure spinor ω_α. Its BRST variation becomes

$$\delta_B \omega_\alpha = \oint d\sigma' \epsilon [\lambda^\beta d_\beta(\sigma'), \omega_\alpha(\sigma)] = \oint d\sigma' \epsilon \lambda^\beta(\sigma')[d_\beta(\sigma'), \omega_\alpha(\sigma)] + \epsilon[\lambda^\beta(\sigma'), \omega_\alpha(\sigma)] d_\beta(\sigma')$$
$$= -(-\epsilon \lambda^\beta \Omega_{\beta\alpha}{}^\gamma)\omega_\gamma + \epsilon d_\alpha,$$

where the first term is a Lorentz rotation.

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Consider now the BRST variation of the gauge current $\overline{J}^I$. It is given by

$$\delta_B \overline{J}^I = \oint d\sigma' \epsilon\lambda^\alpha(\sigma')[d_\alpha(\sigma'), \overline{J}^I(\sigma)] = f^{IJ}{}_K(-\epsilon\lambda^\alpha A_{J\alpha})\overline{J}^K,$$

which is a gauge transformation in the adjoint representation with $-\epsilon\lambda^\alpha A_{J\alpha}$ as gauge parameter.

Now we consider the BRST transformation of Π^A. To obtain them we note that $\delta_B Z^M = \epsilon\lambda^\alpha E_\alpha{}^M$, then

$$\delta_B \Pi^A = \delta_B(\partial Z^M E_M{}^A) = \partial(\delta_B Z^M E_M{}^A) - \delta_B Z^M \partial Z^N \partial_{[N} E_{M]}{}^A,$$

if we use the definition of the torsion, then we have

$$\delta_B \Pi^A = \partial(\epsilon\lambda^\alpha \delta_\alpha^A) + \epsilon\lambda^\alpha \Pi^A \Omega_{A\alpha}{}^A - \epsilon\lambda^\alpha \Pi^B T_{B\alpha}{}^A - \epsilon\lambda^\alpha \Pi^B \Omega_{\alpha B}{}^A (-1)^B.$$

Therefore,

$$\delta_B \Pi^a = \Pi^b(-\epsilon\lambda^\alpha \Omega_{\alpha b}{}^a) - \epsilon\lambda^\alpha \Pi^B T_{B\alpha}{}^a, \qquad \delta_B \Pi^\alpha = \Pi^\beta(-\epsilon\lambda^\gamma \Omega_{\gamma\beta}{}^\alpha) + \nabla(\epsilon\lambda^\alpha) - \lambda^\beta \Pi^B T_{B\beta}{}^\alpha, \tag{3.2}$$

where the first term in each transformation corresponds to a Lorentz rotation and $\nabla(\epsilon\lambda^\alpha) = \partial(\epsilon\lambda^\alpha) + \epsilon\lambda^\beta \Pi^A \Omega_{A\beta}{}^\alpha$. We obtain analogous transformations for $\overline{\Pi}^A$.

The BRST transformation of any background superfield is given by $\delta_B \Psi = \epsilon\lambda^\alpha \partial_\alpha \Psi$. It can be shown that this expression can also be written as a gauge transformation for ψ plus a term which depends on the covariant derivative of the superfield. For example, for the superfield W_I^α one obtains

$$\delta_B W_I^\alpha = W_I^\beta(-\epsilon\lambda^\gamma \Omega_{\gamma\beta}{}^\alpha) - f^{JK}{}_I(-\epsilon\lambda^\beta A_{J\alpha}) W_K^\alpha + \epsilon\lambda^\beta \nabla_\beta W_I^\alpha, \tag{3.3}$$

where the first term is Lorentz rotation of W_I^α and the second is a gauge transformation of W_I^α.

3.1 Nilpotency

Now it will be shown that δ_B^2 acting on the world-sheet fields leads to the nilpotence constraints of (2.4). Consider Z^M first

$$\delta_B^2 Z^M = \delta_B(\epsilon_1 \lambda^\alpha E_\alpha{}^M) = \epsilon_1(\delta_B \lambda^\alpha) E_a{}^M + \epsilon_1 \lambda^\alpha \delta_B E_\alpha{}^M = \epsilon_1\epsilon_2 \lambda^\alpha \lambda^b(-\Omega_{\alpha\beta}{}^\gamma E_\gamma{}^M - \partial_\beta E_\alpha{}^M),$$

by symmetrizing in $(\alpha\beta)$ we form the torsion $T_{\alpha\beta}{}^A E_A{}^M$. Therefore we obtain the constraint $\lambda^\alpha \lambda^\beta T_{\alpha\beta}{}^A = 0$.

Similarly, we compute $\delta_B^2 \lambda^\alpha$

$$\delta_B^2 \lambda^\alpha = \delta_B(-\epsilon_1 \lambda^\beta \lambda^\gamma \Omega_{\gamma\beta}{}^\alpha) = -\epsilon_1\epsilon_2 \lambda^\beta \lambda^\gamma \lambda^\delta (\partial_\delta \Omega_{\gamma\beta}{}^\alpha - \Omega_{\beta\gamma}{}^\sigma \Omega_{\delta\sigma}{}^\alpha - \Omega_{\gamma\delta}{}^\sigma \Omega_{\sigma\beta}{}^\alpha),$$

after symmetrizing in $(\beta\gamma\delta)$ we form the curvature components $R_{\delta\gamma\beta}{}^\alpha$, then we obtain the constraint $\lambda^\beta \lambda^\gamma \lambda^\delta R_{\delta\gamma\beta}{}^\alpha = 0$.

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Now we consider the gauge current $\overline{J}^I$

$$\delta_B^2 \overline{J}^I = \delta_B(-\epsilon_1 f^{IJ}{}_K \lambda^\alpha A_{J\alpha} \overline{J}^K) = -\epsilon_1\epsilon_2 \lambda^\alpha \lambda^\beta f^{IJ}{}_K \overline{J}^K (\partial_\beta A_{J\alpha} - \Omega_{\alpha\beta}{}^\gamma A_{J\gamma}) \\ +\epsilon_1\epsilon_2 \lambda^\alpha\lambda^\beta f^{IJ}{}_K f^{KL}{}_M A_{J\alpha} A_{L\beta} \overline{J}^M,$$

if we symmetrize in $(\alpha\beta)$ and use the fact that the structure constants are the group generators in the adjoint representation, then we can form the field-strength $F_{I\alpha\beta}$ and we obtain the constraint $\lambda^\alpha\lambda^\beta F_{I\alpha\beta} = 0$. It remains to check the nilpotence constraint for the superfield H. For this we need to transform d_α under the pure spinor BRST charge.

3.2 BRST transformation of the superspace translations generator

Now we consider the world-sheet field d_α. Its BRST variation is given by

$$\delta_B d_\alpha = \oint \epsilon(-[\lambda^\beta(\sigma'), d_\alpha(\sigma)] d_\beta(\sigma') + \lambda^\beta(\sigma')[d_\beta(\sigma'), d_\alpha(\sigma)]) \\ = -\epsilon\lambda^\gamma \Omega_{\alpha\gamma}{}^\beta d_\beta + \oint d\sigma' \epsilon \lambda^\beta(\sigma')[d_\beta(\sigma'), d_\alpha(\sigma)],$$

the first term here is not a Lorentz rotation as it was promised. The Lorentz rotation term will appear after the computation of the second term. To do this, we remind the relation between d_α and the remaining world-sheet field (3.1). The more difficult brackets to compute are those coming from the first terms in (3.1). It is due to the fact that there will appear some part integrations to get the right result. After doing the other commutators we obtain

$$\oint d\sigma' \epsilon\lambda^\beta(\sigma')[d_\beta(\sigma'), d_\alpha(\sigma)] = \epsilon\lambda^\beta \Big[-(E_\beta{}^M \partial_M E_\alpha{}^N + E_\alpha{}^M \partial_M E_\beta{}^N) P_N \\ + \overline{J}^I (\partial_{(\alpha} A_{I\beta)} + f^{JK}{}_I A_{J\beta} A_{K\alpha}) \\ + \lambda^\gamma \omega_\delta (\partial_{(\alpha} \Omega_{\beta)\gamma}{}^\delta + \Omega_{\beta\rho}{}^\delta \Omega_{\alpha\gamma}{}^\rho - \Omega_{\beta\gamma}{}^\rho \Omega_{\alpha\rho}{}^\delta) \Big] \\ + \oint d\sigma' \epsilon\lambda^\beta(\sigma') \Big([E_\beta{}^M P_M(\sigma'), \partial_1 Z^N B_{N\alpha}(\sigma)] \\ + [E_\alpha{}^M P_M(\sigma), \partial_1 Z^N B_{N\beta}(\sigma')] \Big).$$

Let us consider the last integral. After doing the commutators we get it is equal to

$$\epsilon\lambda^\beta \Big(-E_\beta{}^M \partial_1 Z^N \partial_M B_{N\alpha} (-1)^{MN} - E_\alpha{}^M \partial_1 Z^N \partial_M B_{N\beta}(-1)^{MN} \Big) \\ - \oint \epsilon\lambda^\beta(\sigma') \Big(E_\beta{}^M(\sigma') B_{M\alpha}(\sigma) \frac{\partial}{\partial\sigma} \delta(\sigma-\sigma') + E_\alpha{}^M(\sigma) B_{M\beta}(\sigma') \frac{\partial}{\partial\sigma'} \delta(\sigma-\sigma') \Big),$$

after integration on σ' we obtain that this expression becomes

$$\epsilon\lambda^\alpha(\partial_1 Z^M E_\beta{}^M \partial_M B_{\alpha N} + \partial_1 Z^M E_\alpha{}^M \partial_M B_{\beta N}) \\ + \epsilon \Big(-(\partial_1\lambda^\beta) B_{\beta\alpha} - \lambda^\beta (\partial_1 E_\beta{}^M) B_{M\alpha} + (\partial_1 \lambda^\beta) B_{\alpha\beta} + \lambda^\beta E_\alpha{}^M \partial_1 B_{M\beta} \Big) = \\ = \epsilon\lambda^\beta((-1)^{M+1} E_\beta{}^M E_\alpha{}^P \partial_{[P} E_{M]}{}^A \partial_1 Z^N B_{NA} + \partial_1 Z^N H_{\beta\alpha N}),$$

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where H stands for the components of the three-form field strength of the two-form superfield B, that is, $H = dB$. Adding up all the contributions we obtain

$$\oint d\sigma' \epsilon[\lambda^\beta d_\beta(\sigma'), d_\alpha(\sigma)] = \epsilon\lambda^\beta \Big[(-1)^{M+1} E_\beta{}^M E_\alpha{}^P \partial_{[P} E_{M]}{}^A (E_A{}^N P_N + \partial_1 Z^N B_{NA}) + \partial_1 Z^N H_{\beta\alpha N} + \overline{J}^I (\partial_{(\alpha} A_{I\beta)} + f^{JK}{}_I A_{J\beta} A_{K\alpha}) + \lambda^\gamma \omega_\delta (\partial_{(\beta} \Omega_{\alpha)\gamma}{}^\delta + \Omega_{\beta\rho}{}^\delta \Omega_{\alpha\gamma}{}^\rho - \Omega_{\beta\gamma}{}^\rho \Omega_{\alpha\rho}{}^\delta)\Big].$$

After using

$$\partial_{(\alpha} A_{I\beta)} + f^{JK}{}_I A_{J\beta} A_{K\alpha} = F_{I\beta\alpha} - (-1)^{M+1} E_\beta{}^M E_\alpha{}^P \partial_{[P} E_{M]}{}^A A_{IA},$$
$$\partial_{(\beta} \Omega_{\alpha)\gamma}{}^\delta + \Omega_{\beta\rho}{}^\delta \Omega_{\alpha\gamma}{}^\rho - \Omega_{\beta\gamma}{}^\rho \Omega_{\alpha\rho}{}^\delta = R_{\beta\alpha\gamma}{}^\delta - (-1)^{M+1} E_\beta{}^M E_\alpha{}^P \partial_{[P} E_{M]}{}^A \Omega_{A\gamma}{}^\delta,$$

(with F is the gauge field-strength , R is the Lorentz curvature) and reminding the definition (2.2) we arrive to the BRST transformation of the world-sheet field d_α to be

$$\delta_B d_\alpha = -\epsilon\lambda^\gamma \Omega_{\alpha\gamma}{}^\beta d_\beta + \epsilon\lambda^\beta (-1)^{M+1} E_\beta{}^M E_\alpha{}^P \partial_{[P} E_{M]}{}^A \left(\frac{1}{2}(\Pi_a + \overline{\Pi}_a)\delta^a_A - \delta^\gamma_A d_\gamma\right) + \epsilon\lambda^\beta \left(\partial_1 Z^N H_{\beta\alpha N} + \overline{J}^I F_{I\beta\alpha} + \lambda^\gamma \omega_\delta R_{\beta\alpha\gamma}{}^\delta\right),$$

we recall that the combination of supervielbein appearing in the second term above is related to the torsion we finally obtain

$$\delta_B d_\alpha = -(-\epsilon\lambda^\gamma \Omega_{\gamma\alpha}{}^\beta) d_\beta + \epsilon\lambda^\gamma d_\beta T_{\gamma\alpha}{}^\beta + \epsilon\lambda^\beta \lambda^\gamma \omega_\delta R_{\alpha\beta\gamma}{}^\delta + \epsilon\lambda^\beta \Pi^a T_{\beta\alpha a}, \tag{3.4}$$

where we recognize a Lorentz rotation in the first term. Here we need that $F_{I\alpha\beta} = H_{\alpha\beta\gamma} = H_{\alpha\beta a} - T_{\alpha\beta a} = 0$ which are consistent with the constrains derived in [10]. In this way the nilpotence constraint for H in (2.4) are satisfied.

In summary, we have proved that the BRST transformations contain a term which corresponds to a gauge and/or Lorentz transformation with field dependent parameters.

4. BRST variation of the action-

As a check we will vary the action (2.1) under the transformations we derived above to derive the holomorphic constraints of (2.5). Before this, let us compute the transformation of the gauge connection $A_I = \Pi^A A_{IA}$ and that of the Lorentz connection $\Omega_\alpha{}^\beta = \Pi^A \Omega_{A\alpha}{}^\beta$ and after that we can deduce analogous transformations for $\overline{A}_I = \overline{\Pi}^A A_{IA}$ and $\overline{\Omega}_\alpha{}^\beta = \overline{\Pi}^A \Omega_{A\alpha}{}^\beta$. These transformations are similar to those of Π^A, the difference is that the result does not contain the torsion but the corresponding curvature. That is, for A_I it will appear the field strength F and for $\Omega_\alpha{}^\beta$ it will appear the curvature R. The result is

$$\delta_B A_I = -\nabla(-\epsilon\lambda^\alpha A_{I\alpha}) - \epsilon\lambda^\alpha \Pi^A F_{IA\alpha}, \qquad \delta_B \Omega_\alpha{}^\beta = -\nabla(-\epsilon\lambda^\gamma \Omega_{\gamma\alpha}{}^\beta) - \epsilon\lambda^\gamma \Pi^A R_{A\gamma\alpha}{}^\beta, \tag{4.1}$$

where we recognize the gauge and Lorentz rotation parts in each transformation.

Since the action is invariant under gauge and Lorentz rotations, we do not need to include that gauge and Lorentz parts in the BRST transformations of the fields appearing in (2.1). Up to gauge and Lorentz transformations the different terms in (2.1) transform in the following way. The variation of the first term is proportional to

$$\int d^2z\epsilon\lambda^\alpha\Pi^{(A}\overline{\Pi}^{a)}T_{A\alpha a}.$$

The variation of the second term of the action is proportional to

$$\int d^2z\epsilon\lambda^\alpha\Pi^A\overline{\Pi}^B H_{BA\alpha},$$

here we have performed integrations by parts and the identity $\overline{\nabla}\Pi^A-\nabla\overline{\Pi}^A=\Pi^B\overline{\Pi}^C T_{CB}{}^A$. The variation of the third term is proportional to

$$\int d^2z-\epsilon\lambda^\alpha\Pi^A\overline{J}^I F_{IA\alpha}.$$

The variation of the fourth term of the action is proportional to

$$\int d^2z\epsilon\left[-d_\alpha\overline{\nabla}\lambda^\alpha-\lambda^\alpha d_\beta\overline{\Pi}^\gamma T_{\alpha\gamma}{}^\beta+\lambda^\alpha\lambda^\beta\omega_\gamma\overline{\Pi}^\delta R_{\alpha\delta\beta}{}^\gamma+\lambda^\alpha\Pi^a\overline{\Pi}^\beta T_{\alpha\beta a}+\lambda^\alpha d_\beta\overline{\Pi}^A T_{A\alpha}{}^\beta\right].$$

The variation of the fifth term is

$$\int d^2z\epsilon\left[\lambda^\alpha d_\beta\overline{J}^I T_{\alpha\gamma}{}^\beta W_I^\gamma+\lambda^\alpha\lambda^\beta\omega_\gamma\overline{J}^I R_{\delta\alpha\beta}{}^\gamma W_I^\delta+\lambda^\alpha\Pi^a\overline{J}^I T_{\alpha\beta a}W_I^\beta-\lambda^\alpha d_\beta\overline{J}^I\nabla_\alpha W_I^\beta\right].$$

The variation of the sixth term plus the $S_{\lambda,\omega}$ is

$$\int d^2z\epsilon\left[d_\alpha\overline{\nabla}\lambda^a-\lambda^\alpha\lambda^\beta\omega_\gamma\overline{\Pi}^A R_{A\alpha\beta}{}^\gamma\right].$$

The variation of the seventh term is

$$\int d^2z\epsilon\left[\lambda^\alpha d_\beta\overline{J}^I U_{I\alpha}{}^\beta+\lambda^\alpha\lambda^\beta\omega_\gamma\overline{J}^I\nabla_\alpha U_{I\beta}{}^\gamma\right].$$

And the variation of $S_{\overline{J}}$ is zero up to gauge transformation.

After summing up all the contributions, we note that the terms involving $\overline{\nabla}\lambda^\alpha$ and $\overline{\Pi}^\alpha$ are zero. Finally the variation of the action becomes

$$\begin{aligned}\delta_B S=\frac{1}{2\pi\alpha'}\int d^2z\epsilon\Big[&\frac{1}{2}\lambda^\alpha\Pi^a\overline{\Pi}^b(T_{\alpha(ab)}+H_{ba\alpha})+\frac{1}{2}\lambda^\alpha\Pi^\beta\overline{\Pi}^a(H_{\beta\alpha a}-T_{\beta\alpha a})+\lambda^\alpha d_\beta\overline{\Pi}^a T_{a\alpha}{}^\beta\\&-\lambda^\alpha\lambda^\beta\omega_\gamma\overline{\Pi}^a R_{a\alpha\beta}{}^\gamma+\lambda^\alpha\Pi^a\overline{J}^I\left(\frac{1}{2}(H_{\alpha\beta a}+T_{\alpha\beta a})W_I^\beta-F_{Ia\alpha}\right)+\\&\lambda^\alpha\Pi^\beta\overline{J}^I\left(\frac{1}{2}H_{\alpha\beta\gamma}W_I^\gamma-F_{I\alpha\beta}\right)+\lambda^\alpha d_\beta\overline{J}^I\left(U_{I\alpha}{}^\beta+T_{\alpha\gamma}{}^\beta W_I^\gamma-\nabla_\alpha W_I^\beta\right)\\&+\lambda^\alpha\lambda^\beta\omega_\gamma\overline{J}^I\left(\nabla_\alpha U_{I\beta}{}^\gamma+R_{\delta\alpha\beta}{}^\gamma W_I^\delta\right)\Big].\end{aligned}\tag{4.2}$$

Therefore, the condition $\delta_B S=0$ determines the classical constraints (2.5) on the background superfields.

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